Quantization of black hole entropy from quasinormal modes

Shao-Wen Wei, Ran Li, Yu-Xiao Liu[1] **and Ji-Rong Ren**

Institute of Theoretical Physics, Lanzhou University,
Lanzhou 730000, P.R. China

E-mail: weishaow06@lzu.cn, liran05@lzu.cn, liuyx@lzu.edu.cn

ABSTRACT: Using the new physical interpretation of quasinormal modes proposed by Maggiore, we calculate the quantum spectra of entropy for various types of non-rotating black holes with no charge. The spectrum is obtained by imposing Bohr-Sommerfeld quantization condition to the adiabatic invariant quantity. We conjecture that the spacing of entropy spectrum is equidistant and is independent of the dimension of spacetime. However, the spacing of area spectrum depends on gravity theory. In Einstein's gravity, it is equally spaced, otherwise it is non-equidistant.

---

[1]Corresponding author.

Contents

1 Introduction

Bekenstein [1] conjectured that, in a quantum gravity theory, the black hole area should be represented by a quantum operator with a discrete spectrum of eigenvalues. By supposing that the black hole horizon area is an adiabatic invariant, he showed that the area spectrum of black hole is equidistant and is of the form

$$\mathcal{A}_n = \epsilon \hbar \cdot n, \quad n = 0, 1, 2, \ldots . \tag{1.1}$$

This rejuvenates the interest of investigation for quantization of black hole area [2–11]. The spacing $\epsilon \hbar$ of the area spectrum has been somewhat controversial. Hod suggested that ϵ can be determined by utilizing the quasinormal mode frequencies of an oscillating black hole [2]. Kunstatter pointed that, for a system with energy E and vibrational frequency $\Delta \omega(E)$, the ratio $\frac{E}{\Delta \omega(E)}$ is an adiabatic invariant. He replaced E with M and identified $\Delta \omega(E)$ as the most appropriate choice for the frequency. So, by way of Bohr-sommerfeld quantization condition, one could derive the spectrum form $\mathcal{A}_n = 4 \hbar \ln 3 \cdot n$, the same with Hod's result. Recently, Maggiore in [13] argued that, in high damping limit, the proper frequency of the equivalent harmonic oscillator, i.e., the quasinormal mode frequencies $\omega(E)$, should be of the form $\omega(E) = \sqrt{|\omega_R|^2 + |\omega_I|^2}$ rather than the real part ω_R. It is clear that when $\omega_I \to 0$, one could get $\omega(E) = \omega_R$ approximately which was adopted extensively in [2, 3, 5]. However, under the case of large n limit or highly excited quasinormal modes for which $\omega_R \ll \omega_I$, the frequency of the harmonic oscillator becomes $\omega(E) = |\omega_I|$.

Motivated by this idea, Medved and Vagenas [8, 9] made choice that the vibrational frequency $\Delta \omega(E) = (|\omega_I|)_m - (|\omega_I|)_{m-1}$ and obtained the area spectrum for kerr black hole. They found there exists a logarithmic term in the adiabatic invariant, which leads to the non-equidistant the area spectrum. In [5], Setare calculated the area spectrum for non-rotating BTZ black hole, and the spectrum is non-equidistant spaced. This result is in contrast with area spectrum of black hole in higher dimension. Here, one may ask what

do other non-rotating black holes with no charge behave and whether the spacing of area or entropy spectrum depends on the dimension of spacetime and gravity theory. In order to answer these questions, we investigate the area and entropy spectra of 3-dimensional non-rotating BTZ black hole, 4-dimensional Schwarzschild black hole and 5-dimensional Gauss-Bonnet (GB) black hole with the choice that $\Delta\omega(E) = (|\omega_I|)_m - (|\omega_I|)_{m-1}$. Our results show that, in Einstein's gravity theory, the spacings of area and entropy spectra are discrete and equally spaced. For Gauss-Bonnet gravity, for example a 5-dimensional Gauss-Bonnet black hole, the entropy spectrum is still discrete and equally spaced but the spacing of area spectrum is not equidistant. However, when setting the Gauss-Bonnet coupling constant $\alpha_{GB} \to 0$, the spacing of area spectrum becomes equidistant. In summary, for non-rotating black holes with zero charge, we conjecture that the spacing of entropy spectrum is equidistant and independent of the dimension of spacetime and gravity theory. But the spacing of area spectrum depends on gravity theory. In Einstein's gravity, it is equally spaced, otherwise it is non-equidistant.

This paper is organized as follows. In section 2, we briefly review the method used in this paper and obtain the entropy quantization of Schwarzschild black hole. The discussions for 3-dimensional non-rotating BTZ black hole and 5-dimensional Gauss-Bonnet black hole appear in sections 3 and 4. Finally, the paper ends with a brief conclusion.

2 Entropy quantization of 4-dimensional Schwarzschild black hole

In this section, we modify the frequency that appears in the adiabatic invariant. Then through calculating the adiabatic invariant of Schwarzschild black hole, we manage to obtain the entropy and area spectrum.

Firstly, we consider 4-dimensional Schwarzschild black hole, which is charactered by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\Omega_2, \tag{2.1}$$

where M is the mass of black hole. The radius of the event horizon r_h is $r_h = 2M$. The surface gravity $\kappa = \frac{1}{4M}$. Area and Hawking temperature for this black hole are given by

$$\mathcal{A} = 4\pi r_h^2 = 16\pi M^2 \tag{2.2}$$

and

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, \tag{2.3}$$

respectively.

In ref. [2], Hod succeeded in deriving the quantum of the area spectrum using the Bohr's Correspondence principle from quasinormal modes. The complex quasinormal modes that correspond to the perturbation equation of Schwarzschild black hole are also obtained:

$$M\omega_m = 0.0437123 - \frac{i}{4}\left(m + \frac{1}{2}\right) + \mathcal{O}\left[(m+1)^{-\frac{1}{2}}\right]. \tag{2.4}$$

Noting that the highly damped ringing frequencies depend only upon the black hole mass. This feature is consistent with the interpretation of the highly damped ringing frequencies as characteristics of the black hole itself in the $m \gg 1$ limit.

Kunstatter proposed that given a system with energy E and vibrational frequency $\Delta\omega(E)$, a natural adiabatic invariant quantity is [12]:

$$I = \int \frac{dE}{\Delta\omega(E)}. \tag{2.5}$$

In the large n limit, the Bohr-Sommerfeld quantization can be expressed as

$$I = n\hbar. \tag{2.6}$$

Making the choice $\Delta\omega(E) \approx \frac{\ln 3}{8\pi M}$, one can obtain the area spectrum of a Schwarzschild black hole

$$\mathcal{A}_n = 4\ell_p^2 \ln 3 \cdot n, \tag{2.7}$$

where ℓ_p is the Planck length. It can be seen that the area spectrum is equally spaced with spacing $4\ell_p^2 \ln 3$ and in agree with the Bekenstein's conjecture.

Recently, Maggiore refined Hod's treatment by arguing that the physically relevant frequency would actually be [13]

$$\omega(E) = \sqrt{|\omega_R|^2 + |\omega_I|^2}, \tag{2.8}$$

where ω_R and ω_I are the real and imaginary parts of the quasinormal mode frequency respectively. When $\omega_I \to 0$, one could get $\omega(E) = |\omega_R|$ approximately. However, under the case of large m or highly excited quasinormal modes for which $\omega_R \ll \omega_I$, the frequency of the harmonic oscillator becomes $\omega(E) = |\omega_I|$. With this supposition, Vagenas and Medved obtained the area spectrum of Kerr black hole. They calculated the adiabatic invariant quantity I and area spectrum, which are given by [8, 9]

$$I = \frac{\mathcal{A}}{4\pi} - 2M^2 \log\left(\frac{\mathcal{A}}{8\pi}\right), \tag{2.9}$$

$$\mathcal{A}_n + \mathcal{O}(J_n^4) = 4\pi\ell_{Pl}^2 \cdot n. \tag{2.10}$$

This area spectrum is non-equidistant and depends on the angular momentum of Kerr black hole. It is shown that, when the angular momentum J is small, the area spectrum is equidistant.

Here, we want to ask what will happen for Schwarzschild black hole under this supposition. Now, let us turn back to the Schwarzschild black hole. The adiabatic invariant quantity I for Schwarzschild black hole is

$$I = \int \frac{dM}{\Delta\omega}. \tag{2.11}$$

From the quasinormal modes (2.4), vibrational frequency $\Delta\omega$ can be obtained

$$\Delta\omega = (|\omega_I|)_m - (|\omega_I|)_{m-1} = \frac{1}{4M}. \tag{2.12}$$

Substituting it into (2.11) and using the Bohr-Sommerfeld quantization condition (2.6), one can obtain the area spectrum for Schwarzschild black hole

$$\mathcal{A}_n = 8\pi\ell_{Pl}^2 \cdot n, \tag{2.13}$$

which precisely coincides with Bekenstein's result. The entropy spectrum can be obtained from the relation $S = \frac{A}{4}$. The entropy spectrum is given by

$$S_n = 2\pi \ell_{Pl}^2 \cdot n, \tag{2.14}$$

which is an equidistant spectrum.

3 Entropy quantization of $(2+1)$-dimensional non-rotating BTZ black hole

Now, in this section, we would like to apply the method to deal with a (2+1)-dimensional non-rotating BTZ black hole. The line element for non-rotating BTZ black hole is

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2}\right)} + r^2 d\theta^2, \tag{3.1}$$

where M is the Arnowitt-Deser-Misner (ADM) mass and cosmological constant is given by $\Lambda = \frac{1}{\ell^2}$. The event horizon locates at

$$r_h = \sqrt{\frac{M}{\Lambda}}, \tag{3.2}$$

and its area is given by

$$\mathcal{A} = 2\pi r_h = 2\pi\sqrt{\frac{M}{\Lambda}}. \tag{3.3}$$

The quasinormal frequency for non-rotating BTZ black hole has been obtained by Cardoso and Lemos in [14]

$$\omega = \pm m - 2iM^{1/2}(m+1), \quad m = 0, 1, 2, \dots . \tag{3.4}$$

The adiabatic invariant quantity I for non-rotating BTZ black hole is

$$I = \int \frac{dM}{\Delta\omega}. \tag{3.5}$$

At large m, the vibrational frequency $\Delta\omega$ is

$$\Delta\omega = (|\omega_I|)_m - (|\omega_I|)_{m-1} = 2\sqrt{M}. \tag{3.6}$$

Substituting $\Delta\omega$ into (3.5), we obtain the adiabatic invariant quantity

$$I = \sqrt{M}. \tag{3.7}$$

Using Bohr-Sommerfeld quantization condition (2.6), we have

$$\sqrt{M} = n\hbar. \tag{3.8}$$

Recalling the area from (3.3), we derive the area spectrum of this black hole

$$\mathcal{A}_n = 2\pi n\hbar\sqrt{\frac{1}{\Lambda}}. \tag{3.9}$$

It is clear that the cosmological constant Λ appears in the area spectrum (3.9). We also note that this area spectrum is equally spaced. In ref. [5], the result is $\mathcal{A}_n = 2\pi\sqrt{\frac{nm\hbar}{\Lambda}}$. Although this spectrum is quantized, it is not equally spaced. This conflicts with the conjecture of Bekenstein. In ref. [13] Maggiore argues that, under high damped modes, it is not accurate to make $\omega = \omega_R$. For this case, one needs take $\omega = (\omega_R^2 + \omega_I^2)^{\frac{1}{2}}$. Through taking the vibrational frequency $\Delta\omega = (|\omega_I|)_m - (|\omega_I|)_{m-1}$, we get the area spectrum (3.9), which has equidistant $\Delta\mathcal{A} = 2\pi\hbar\sqrt{\frac{1}{\Lambda}}$. Entropy spectrum of this black hole is

$$S_n = \frac{1}{2}\pi\hbar\sqrt{\frac{1}{\Lambda}} \cdot n, \tag{3.10}$$

with the spacing

$$\Delta S = S_{n+1} - S_n = \frac{1}{2}\pi\hbar\sqrt{\frac{1}{\Lambda}}. \tag{3.11}$$

It is clear that this entropy is equidistant and the spacing depends on the cosmological constant Λ.

4 Entropy quantization of 5-dimensional Gauss-Bonnet black hole

In this section, we investigate the entropy and area quantization of black hole in Gauss-Bonnet gravity theory. The (4+1) dimensional static, spherically symmetric black hole solution in this theory is of the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3, \tag{4.1}$$

where the metric function is

$$f(r) = 1 + \frac{r^2}{2\alpha}\left[1 - \left(1 + \frac{4\,\alpha\,\varpi}{r^4}\right)^{1/2}\right]. \tag{4.2}$$

Here, $\alpha = 2\alpha_{GB}$ and ϖ is related to the ADM mass M by the relationship $\varpi = \frac{16\pi}{3\Sigma_3}\,M$, where Σ_3 is the volume of unit 3 sphere. The event horizon is located at $r = r_h$, and r_h satisfies

$$r_h^2 + \alpha - \varpi = 0. \tag{4.3}$$

For the horizon to exist at all, one must have $r_h^2 + 2\alpha \geq 0$.

Area and Hawking temperature for this black hole are given, respectively, by

$$\mathcal{A} = 2\pi^2\left(-\alpha + \frac{8M}{3\pi}\right)^{3/2}, \tag{4.4}$$

and

$$T = \frac{1}{2\pi}\frac{r_h}{(r_h^2 + 2\alpha)}. \tag{4.5}$$

The highly damped quasinormal modes for 5-dimensional GB black hole (when $\omega_I \gg \omega_R$) have been worked out [15]

$$\omega(m) \xrightarrow[m\to\infty]{} T\ln Q + i(2\pi T)m. \tag{4.6}$$

The imaginary part can be understood in terms of a scattering matrix formalism; see e.g., [16]. We identify the relevant frequency as $\omega = |\omega_I|$. The adiabatic invariant quantity I for this black hole is

$$I = \int \frac{dM}{\Delta\omega}. \tag{4.7}$$

From the quasinormal modes (4.6), vibrational frequency $\Delta\omega$ can be evaluated as

$$\Delta\omega = (|\omega_I|)_m - (|\omega_I|)_{m-1} = 2\pi T. \tag{4.8}$$

Using the expression of event horizon area for GB black hole, the adiabatic invariant is rewritten as

$$I = \frac{8M + 15\pi\alpha}{12}\sqrt{\frac{8M}{3} - \alpha}. \tag{4.9}$$

With Bohr-Sommerfeld quantization condition (2.6) and the area (4.4), we obtain the area spectrum

$$\mathcal{A}_n + 6\alpha(4\pi^4\mathcal{A}_n)^{\frac{1}{3}} = 8\pi\hbar \cdot n. \tag{4.10}$$

The spacing of this spectrum is

$$\Delta\mathcal{A} = \mathcal{A}_{n+1} - \mathcal{A}_n = 8\pi\hbar + g(\mathcal{A}_{n,n+1}), \tag{4.11}$$

where, the function $g(\mathcal{A}_{n,n+1})$ is

$$g(\mathcal{A}_{n,n+1}) = 6\alpha(4\pi^4)^{\frac{1}{3}}(\mathcal{A}_n^{\frac{1}{3}} - \mathcal{A}_{n+1}^{\frac{1}{3}}). \tag{4.12}$$

The function $g(\mathcal{A}_{n,n+1})$ is a correction term to the spacing of the spectrum, and leads to a non-equidistant area spectrum.

In Einstein's gravity, entropy of the horizon is proportional to its area. Equidistance of the area spectrum implies that the entropy spectrum is also equidistant. However, when one considers the natural generalization of Einstein gravity by including higher derivative correction terms like the Gauss-Bonnet term to the original Einstein-Hilbert action, the trivial relationship $S = \frac{A}{4}$ between horizon area and associated entropy does not hold anymore. The relationship is now

$$S = \frac{\mathcal{A}}{4}\left[1 + 6\alpha\left(\frac{\mathcal{A}}{\Sigma_3}\right)^{-2/3}\right]. \tag{4.13}$$

Substituting (4.10) into (4.13), we obtain the entropy spectrum of GB black hole

$$S_n = 2\pi\hbar \cdot n. \tag{4.14}$$

It is clear that this entropy spectrum is equally spaced with $\Delta S = 2\pi\hbar$. Our results are in agreement with that of Kothawala et al. [10], where the entropy eigenvalues of GB black hole are discrete and equally spaced, but the area spacing is not equidistant. So one can concluded that, for GB gravity, the notion of quantum of entropy is more appropriate than the quantum of area. One can see that, the area spectra for 3-dimensional non-rotating BTZ black and 4-dimensional Schwarzschild black hole are all discrete and equally spaced.

However, these characters partially hold for 5-dimensional GB black hole. It seems that, whether the area spectrum is equidistant does not depend on the dimension of spacetime, but depend on gravity theory. It is easy to see that, if we set the Gauss-Bonnet coupling constant α_{GB} to 0, the equation (4.13) shows a linear relationship between entropy and area. Then equidistant entropy spectrum implies the equidistant area spectrum. The equidistant area spectrum also can be seen from function $g(\mathcal{A}_{n,n+1}) = 0$ when Gauss-Bonnet coupling constant $\alpha_{GB} \to 0$.

5 Conclusion

In summary, by modifying the frequency that appears in the adiabatic invariant of black hole and using the Bohr-Sommerfeld quantization under large n limit, we investigate the entropy and area spectra of Schwarzschild black hole, (2+1)-dimensional non-rotating BTZ black hole and 5-dimensional Gauss-Bonnet black hole, respectively. All these results imply that the entropy for different types of black holes can be quantized and equally spaced. The area can also be quantized, but the spacing depends on the gravity theory. In Einstein's gravity, area spectrum is equally spaced, but in Gauss-Bonnet gravity, the spacing of the area spectrum is non-equidistant. Furthermore, for the non-rotating BTZ black hole, the spacings of both entropy and area spectra depend on the cosmological constat Λ. So this result may imply some intrinsic characteristics of the non-rotating BTZ black hole. It is worth to point out that the result we obtained is only for non-rotating black holes with no charge. For more general cases, the investigation should be carried out in our further work. We should keep in mind that, all our's calculations are semiclassical and based on Bohr-Sommerfeld quantization condition and the quasinormal modes. These results are just conjectural and the underlying relationship between entropy quantization and quasinormal modes should be given more attention.

Acknowledgments

References

[1] J.D. Bekenstein, *The quantum mass spectrum of the Kerr black hole, Lett. Nuovo Cim.* **11** (1974) 467 [SPIRES]; *Quantum black holes as atoms,* gr-qc/9710076 [SPIRES].

[2] S. Hod, *Bohr's correspondence principle and the area spectrum of quantum black holes,* *Phys. Rev. Lett.* **81** (1998) 4293 [gr-qc/9812002] [SPIRES]; *Best approximation to a reversible process in black-hole physics and the area spectrum of spherical black holes,* *Phys. Rev.* **D 59** (1999) 024014 [gr-qc/9906004] [SPIRES].

[3] O. Dreyer, *Quasinormal modes, the area spectrum and black hole entropy,* *Phys. Rev. Lett.* **90** (2003) 081301 [gr-qc/0211076] [SPIRES].

[4] A.P. Polychronakos, *Area spectrum and quasinormal modes of black holes*, *Phys. Rev.* **D 69** (2004) 044010 [hep-th/0304135] [SPIRES].

[5] M.R. Setare, *Non-rotating BTZ black hole area spectrum from quasi-normal modes*, *Class. Quant. Grav.* **21** (2004) 1453 [hep-th/0311221] [SPIRES].

[6] M.R. Setare, *Area spectrum of extremal Reissner-Nordström black holes from quasi-normal modes*, *Phys. Rev.* **D 69** (2004) 044016 [hep-th/0312061] [SPIRES]; *Near extremal Schwarzschild-de Sitter black hole area spectrum from quasi-normal modes*, *Gen. Rel. Grav.* **37** (2005) 1411 [hep-th/0401063] [SPIRES]; M.R. Setare and E.C. Vagenas, *Area spectrum of Kerr and extremal Kerr black holes from quasinormal modes*, *Mod. Phys. Lett.* **A 20** (2005) 1923 [hep-th/0401187] [SPIRES]; M.R. Setare, *Area spectrum of near extremal black branes from quasi-normal modes*, *Int. J. Theor. Phys.* **44** (2005) 1365 [hep-th/0504015] [SPIRES].

[7] S. Lepe and J. Saavedra, *Quasinormal modes, superradiance and area spectrum for* $2 + 1$ *acoustic black holes*, *Phys. Lett.* **B 617** (2005) 174 [gr-qc/0410074] [SPIRES].

[8] E.C. Vagenas, *Area spectrum of rotating black holes via the new interpretation of quasinormal modes*, *JHEP* **11** (2008) 073 [arXiv:0804.3264] [SPIRES].

[9] A.J.M. Medved, *On the Kerr quantum area spectrum*, *Class. Quant. Grav.* **25** (2008) 205014 [arXiv:0804.4346] [SPIRES].

[10] D. Kothawala, T. Padmanabhan and S. Sarkar, *Is gravitational entropy quantized?*, *Phys. Rev.* **D 78** (2008) 104018 [arXiv:0807.1481] [SPIRES].

[11] T. Tanaka and T. Tamaki, *Black hole entropy for the general area spectrum*, arXiv:0808.4056 [SPIRES].

[12] G. Kunstatter, *d-dimensional black hole entropy spectrum from quasi-normal modes*, *Phys. Rev. Lett.* **90** (2003) 161301 [gr-qc/0212014] [SPIRES].

[13] M. Maggiore, *The physical interpretation of the spectrum of black hole quasinormal modes*, *Phys. Rev. Lett.* **100** (2008) 141301 [arXiv:0711.3145] [SPIRES].

[14] V. Cardoso and J.P.S. Lemos, *Scalar, electromagnetic and Weyl perturbations of BTZ black holes: quasi normal modes*, *Phys. Rev.* **D 63** (2001) 124015 [gr-qc/0101052] [SPIRES].

[15] R.G. Daghigh, G. Kunstatter and J. Ziprick, *The mystery of the asymptotic quasinormal modes of Gauss-Bonnet black holes*, *Class. Quant. Grav.* **24** (2007) 1981 [gr-qc/0611139] [SPIRES]; S.K. Chakrabarti and K.S. Gupta, *Asymptotic quasinormal modes of d-dimensional Schwarzschild black hole with Gauss-Bonnet correction*, *Int. J. Mod. Phys.* **A 21** (2006) 3565 [hep-th/0506133] [SPIRES].

[16] T. Padmanabhan, *Quasi normal modes: a simple derivation of the level spacing of the frequencies*, *Class. Quant. Grav.* **21** (2004) L1 [gr-qc/0310027] [SPIRES]; T.R. Choudhury and T. Padmanabhan, *Quasi normal modes in Schwarzschild-deSitter spacetime: a simple derivation of the level spacing of the frequencies*, *Phys. Rev.* **D 69** (2004) 064033 [gr-qc/0311064] [SPIRES].